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Vibrating giant spikes and the large-winding sector

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Abstract: The single spike is a rigidly rotating classical string configuration closely related to the giant magnon. We calculate bosonic and fermionic modes of this solution, from which we see that it is not supersymmetric. It can be viewed as an excitation above a hoop of string wound around the equator, in the same sense that the magnon is an excitation above an orbiting point particle. We find the operator which plays the role of the Hamiltonian for this sector, which compared to the magnon's $\Delta - J$ has the angular momentum replaced by a winding charge. The single spike solution is unstable, and we use the modes to attempt a semi-classical computation of its lifetime.

Keywords: AdS-CFT Correspondence, Solitons Monopoles and Instantons, Integrable Field Theories.

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Contents

1. Introduction

Much has been learned about the AdS-CFT correspondence, which relates large-N gauge theory to string theory [1], by looking at limits in which an SO(6) charge J also becomes large. At large λ the theory is a theory of classical strings moving in $AdS_5 \times S^5$, with J an angular momentum on the sphere, while at small λ it is perturbative Yang-Mills theory in 4 dimensions, with J an R-charge of this theory. [2, 3] This is the large-J sector of the correspondence.

The first well-studied example in this sector is the BMN limit [4, 5] which on the string side, consists of nearly point-like solutions orbiting the sphere, experiencing a pp-wave geometry. On the gauge theory side, the anomalous dimension $\Delta - J$ can be computed as the energy of a ferromagnetic spin chain. [6 – 8] These spin chains are integrable systems, allowing the use of Bethe ansatz techniques to compute the spectrum from the S-matrix for two-particle scattering. [9 – 14] (In some cases one can explicitly recover the string action from the spin-chain [15].)

The elementary excitations of spin chains are magnons, which to be scattered must have some momentum $p \neq 0$. Extending the theory to allow lone magnons with momentum leads to the centrally extended algebras [8, 16] on the gauge side, dual to strings which do not close. These are the giant magnons [6]. Generalisations which have been explored include magnons with more than one large angular momentum [17 – 19] and magnons with finite J [20 – 23].

Giant magnons are one type of rigidly rotating strings with cusps, moving on the sphere and made as large as they can be. In general these are called spiky strings, and they also exist in flat space [24, 25] and in AdS. [26, 27] In flat space T-duality leads to another class of spiky strings, with cusps pointing inwards, and these 'T-dual' solutions can also exist on the sphere. Starting with one of these and taking the same maximum-size limit used for the magnon then leads to the single spike solution which we study here. [28] Recent papers on the single spike include [29 – 35].

The giant magnon can be viewed as an excitation above a vacuum solution of a point particle orbiting along the equator. [36] (The label 'giant' is meant to indicate that they explore much of the S^5 geometry, as the earlier giant gravitons did. [37, 38]) Fluctuations of this vacuum have Hamiltonian $\Delta - J$ [39] (where $J \leq \Delta$ is the BPS bound.) The single spike is similarly an excitation above a string wound around the equator, which we call the hoop. In the Hamiltonian for fluctuations, the angular momentum J is replaced with a measure of the winding along the same direction, which we call Φ. This is almost T-duality, except that the circle involved is part of sphere. It is not clear whether this duality can be usefully related to the T-duality used in [40] and [41], in S^5 and AdS.

The single spike, and indeed the hoop, are not supersymmetric. Exploring the correspondence in sectors with less or no supersymmetry is of great interest, and it is our hope that the close relationship to the magnon case can be used as a tool for this. The gauge theory dual of the single spike is not known, but it is conjectured to be some excitation of an anti-ferromagnetic state of the spin chain [42, 43] in what has been named the large-winding sector of the correspondence [44]. In the absence of supersymmetry it is possible that integrability will help to find the dual of the spike solution.

Solitons have long been studied in field theory, and a set of tools called semi-classical quantisation enables us to learn about the related objects in the quantum theory. [45 – 51] Many of these techniques have been revived to study solutions of classical string theory in $AdS_5 \times S^5$ [39, 52 – 54] (which is known to be integrable [55, 56]). We have the extra complication that the single spike is an excitation of an unstable vacuum state (as the string wrapped around an equator of S^5 can slide off towards the pole) so what we aim to calculate by these methods is not an energy correction but a lifetime, as discussed in the text.

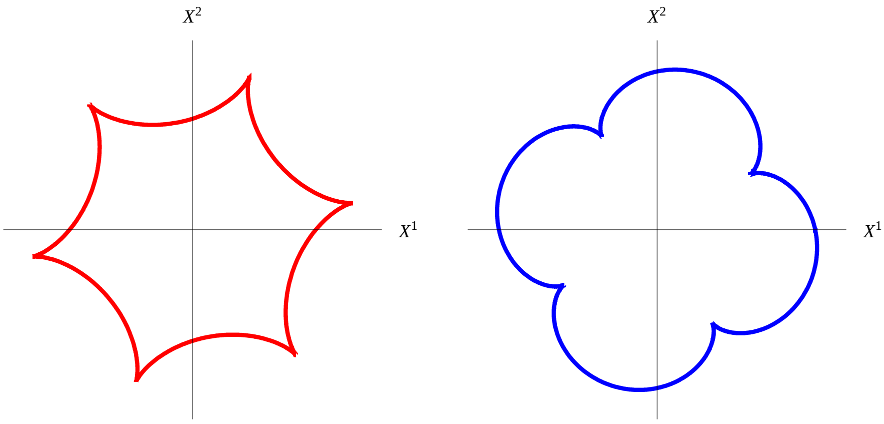

Figure 1: The original and T-dual spiky string in flat space. Both are drawn with $B = 5$, leading to 6 and 4 spikes respectively.

In section 2 we set out the solution, and write down its bosonic modes. The bosonic string on the sphere can be mapped to sine-gordon theory, where the giant magnon becomes the simple kink. The single spike is instead mapped to an unstable kink.

Section 3 contains the calculation of the ferminonic modes, along the lines of what was done for the giant magnon in [57] and [53]. We find that, compared to the magnon, some of the bosonic modes become tachyonic, while the fermionic modes double in number and become massless, ruling out supersymmetry.

In section 4 we study the vacuum of the large-winding sector, the infinitely wound hoop. We find that $\Delta - \Phi$ is the Hamiltonian for perturbations of this hoop, and therefore is the charge which receives quantum corrections. After making a rough calculation of these corrections we conclude in section 5.

Appendix A is a computation of fermionic zero modes, which are in this case the $\omega \to 0$ end of the continuum of non-zero modes. Appendix B has some further details on quantum corrections.

2. Bosonic sector

2.1 Spiky strings in flat space

The spiky string in flat space is the solution [26, 24, 25]

$$\begin{aligned} X^0 &= t, \\ X^1 &= A\cos\left(\frac{t+x}{2A}\right) + AB\cos\left(\frac{t-x}{2AB}\right), \\ X^2 &= A\sin\left(\frac{t+x}{2A}\right) + AB\sin\left(\frac{t-x}{2AB}\right), \end{aligned} \tag{2.1}$$

with two parameters A, B. This describes a rigidly rotating string with $n = B + 1$ cusps, or spikes, pointing outwards (see figure 1). A clearly controls the overall size.

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Since this solution has neither centre-of-mass momentum nor winding, the effect of T-duality in the X^2 direction is to change the sign of the left-movers in that direction, [28] giving

$$\begin{aligned} &X^0 \text{ and } X^1 \text{ unchanged}, \\ &X^2 = A \sin\left(\frac{t+x}{2A}\right) - AB \sin\left(\frac{t-x}{2AB}\right), \end{aligned} \tag{2.2}$$

which is another rigidly rotating string, now with $B-1$ spikes pointing inwards. In both cases the cusps always move at the speed of light.

Notice that the T-dual solution could be obtained by simply interchanging x and t in the spatial X^i. This symmetry is visible in the equations of motion

$$\left(-\partial_t^2 + \partial_x^2\right) X^i = 0$$

and in the Virasoro constraints (for $X^0 = t$)

$$\left(\partial_t X^i\right)^2 + \left(\partial_x X^i\right)^2 = 1, \qquad \partial_t X^i \partial_x X^i = 0,$$

all of which are unchanged by $x \leftrightarrow t$.

2.2 On the sphere

Similar solutions exist on the sphere, and when they are small will reduce to those in flat space. In [28] it was shown that if the analogue of the original solution (2.1) becomes large, so that the spikes touch the equator, then each segment (between spikes) of it becomes a giant magnon. For the analogue of the T-dual solution (2.2), the limit in which the lobes touch the equator is the single spike which this paper studies.

We embed the sphere in $\mathbb{R}^6$, parameterised by X^i (with $X^i X^i = 1$), and look at solutions rotating in the $Z_1 = X^1 + iX^2$ plane. The remaining four directions we call $\vec{X}$, and X^0 is the time co-ordinate (ultimately from AdS).

The giant magnon [6] is the following solution:

$$\begin{aligned} X^0 &= t, \\ Z_1 &= e^{it}\left(c + i\sqrt{1-c^2}\tanh u\right), \\ \vec{X} &= \vec{n}\sqrt{1-c^2}\,\mathrm{sech}\, u, \end{aligned} \tag{2.3}$$

where we write $c = \cos(p/2)$ for the worldsheet velocity, and (u, v) are boosted worldsheet co-ordinates

$$\begin{aligned} u &= \gamma(x - ct), \\ v &= \gamma(t - cx), \qquad \text{with } \gamma = \frac{1}{\sqrt{1-c^2}} = \frac{1}{\sin(p/2)}. \end{aligned} \tag{2.4}$$

Note all of $-\infty < x < \infty$ covers only one of the curves between cusps. It is understood that the physical closed-string solution consists of several giant magnons connected together.

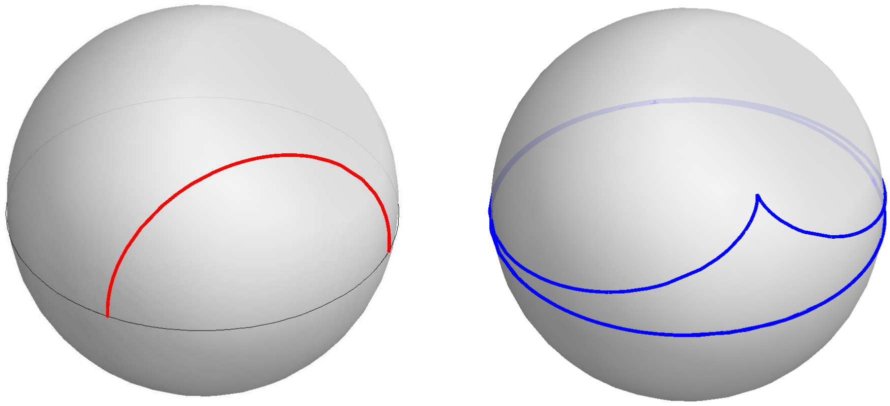

Figure 2: The giant magnon (left, $c = \cos(p/2) = 0.7$) and the single spike (right, $c = 0.8$). These are both are rigidly rotating along the equator, with their cusps moving at the speed of light.

The case $c = 0$ (zero worldsheet velocity, $p = \pi$) is one of GKP's folded strings. [2] In the limit $p \to 0$ the magnon becomes a point particle moving along the equator.

This solution is written in conformal gauge (i.e. the induced metric is proportional to the standard metric, $\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \propto \eta_{ab}$) and thus solves the Virasoro constraints

$$\left(\partial_t X^i\right)^2 + \left(\partial_x X^i\right)^2 = 1, \qquad \partial_t X^i \partial_x X^i = 0,$$

and the conformal gauge equations of motion

$$\left(-\partial_t^2 + \partial_x^2\right) X^i + X^i \left(-(\partial_t X^j)^2 + (\partial_x X^j)^2\right) = 0.$$

(The extra term, compared to flat space, comes from the constraint $X^i X^i = 1$.) As in flat space, these are unchanged by the interchange of x and t. So there is another solution $X^i_{\text{spike}}(t,x) = X^i_{\text{magnon}}(x,t)$, which has been dubbed the single spike: [28]

$$\begin{aligned} X^0 &= t, \\ Z_1 &= e^{ix}\left(c + i\sqrt{1-c^2}\tanh v\right), \\ \vec{X} &= \vec{n}\sqrt{1-c^2}\,\operatorname{sech} v\,. \end{aligned} \tag{2.5}$$

This solution is drawn in figure 2. We keep the same parameter c, although the worldsheet velocity is now $1/c$ in the x, t co-ordinates.[1] For the most part we will be interested only in the range $0 < c < 1$.

Both solutions are localised on the worldsheet. As $x \to \infty$, the magnon solution approaches the point particle $Z_1 = e^{it}$ and $\vec{X} = 0$ while the single spike solution becomes instead the infinitely wound hoop $Z_1 = e^{ix}$. The point particle and the hoop are clearly related by the same $x \leftrightarrow t$ swop, and they are also the vacuum solutions needed to obtain the magnon or the single spike by the dressing method, which survives this interchange. [58, 59, 36, 31]

[1] This is related to the parameter θ_0 used in [28], which is the angle from the north pole to the tip of the spike, by $\sin\theta_0 = c = \cos(p/2)$. Also note that $\overline{\theta} = \frac{\pi}{2} - \theta_0 = p/2$.

The string's conserved charges of interest are defined as

$$\begin{aligned}
\Delta &= \frac{\sqrt{\lambda}}{2\pi}\int dx\, 1 && \text{energy (simple in this gauge!)}, \\
J &= \frac{\sqrt{\lambda}}{2\pi}\int dx\ \mathrm{Im}\left(\overline{Z}_1 \partial_t Z_1\right) && \text{angular momentum in } Z_1 \text{ plane}, \\
\Phi &= \frac{\sqrt{\lambda}}{2\pi}\int dx\ \mathrm{Im}\left(\partial_x \log Z_1\right) = \frac{\sqrt{\lambda}}{2\pi}\Delta\phi && \text{winding charge.}
\end{aligned} \tag{2.6}$$

This Φ is a conveniently scaled version of the the opening angle $\Delta\phi$, where $\phi = \arg Z_1$ is the azimuthal angle.

For the magnon, Δ and J are infinite, and we have the familiar

$$\begin{aligned}
\Delta - J &= \frac{\sqrt{\lambda}}{\pi}\sin(p/2), \\
\Phi &= \frac{\sqrt{\lambda}}{2\pi}p\,.
\end{aligned} \tag{2.7}$$

For the single spike, it is Φ instead of J which is infinite, and

$$\begin{aligned}
\Delta - \Phi &= \frac{\sqrt{\lambda}}{2\pi}p, \\
J &= \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\,.
\end{aligned} \tag{2.8}$$

2.3 Zero modes

Bosonic zero modes are the variations produced by changing collective co-ordinates:

$$\delta_v X^i = -\frac{\partial X^i}{\partial v_0}\Big|_{v_0=0},$$

where v_0 is some modulus. Writing the single spike solution (2.5) with explicit x_0 and v_0 in addition to the direction $\vec{n}$

$$\begin{aligned}
Z_1 &= e^{i(x-x_0)}\left(c + i\sqrt{1-c^2}\tanh(v-v_0)\right), \\
\vec{X} &= \vec{n}\sqrt{1-c^2}\,\mathrm{sech}(v-v_0),
\end{aligned}$$

we obtain the following modes:

- δ_x, a rigid rotation of Z_1:

$$\begin{aligned}
\delta_x Z_1 &= iZ_1\,, \\
\delta_x \vec{X} &= 0\,;
\end{aligned}$$

- δ_v, a reparametrisation along v:

$$\begin{aligned}
\delta_v Z_1 &= e^{it} i\sqrt{1-c^2}\,\mathrm{sech}^2 v\,, \\
\delta_v \vec{X} &= -\vec{n}\sqrt{1-c^2}\,\mathrm{sech}\, v\ \tanh v\,;
\end{aligned} \tag{2.9}$$

- δ_m, a rotation of the orientation $\vec{n}$:

$$\delta_m Z_1 = 0\,,$$
$$\delta_m \vec{X} = \vec{m}\,\sqrt{1-c^2}\,\mathrm{sech}\, v\,,$$

where $\vec{m}\cdot\vec{n}=0$, thus there are three such modes.

It may seem strange to work out δ_v holding x fixed (and δ_x holding v fixed) rather than always using one pair x, t or u, v. This simply gives a convenient linear combination of the modes, in which one is normalisable and one is not. If instead we worked out δ_t holding x fixed (and *vice versa*) we would get

$$\delta_{t|x} X^i = \gamma \delta_v X^i \qquad\qquad \delta_{x|t} X^i = \delta_x X^i - c\gamma\delta_v X^i$$
$$\delta_{t|x} X^0 = 1 \qquad\qquad \delta_{x|t} X^0 = 0$$

where we now write the time components in addition to the spatial ones. In spacetime the meaning of these two modes ($\delta_{t|x}$ and $\delta_{x|t}$) is clear: at any point they are the two tangent vectors to the string. In fact they are exactly the co-ordinate basis vectors from x, t. These would normally generate reparametrisations, not physical modes.

But here, as for the giant magnon, we are not studying the complete string solution, but rather just a section of it.[2] There must be solitons elsewhere on the worldsheet, and motion relative to these is physical. Thus we keep one of these modes, along with the 3 modes δ_m, making 4 zero modes in total.[3]

Notice that the other physical zero modes, the perpendicular rotations δ_m, are independent of u. The corresponding modes in the magnon case, (2.15) in [53], are independent of v, which is time boosted by c. This is a reason for regarding u as being the time co-ordinate for the purpose of identifying zero and non-zero modes.[4]

2.4 Non-zero modes

Inserting $X^i + \delta X^i$ into the equation of motion, we obtain the equation for the fluctuations

$$\partial_a \partial^a \delta X^i + \left(1 - 2\,\mathrm{sech}^2\, v\right)\delta X^i - \left(X^j \partial_a \partial^a \delta X^j\right) X^i = 0.$$

The zero modes above are solutions of this equation. We now seek non-zero modes, i.e. solutions of the form

$$\delta X^j = e^{ikv - i\omega u}\, f^j(v).$$

[2] We return to this question in section 4.3 below.

[3] Our mode δ_v (2.9) is the analogue of [57]'s (3.11) and [53]'s (2.16). In [57] this is derived from a translation of the sine-gordon soliton.

[4] We could regard u as being the product of a boost by velocity $\frac{1}{c} > 1$:

$$u = \gamma(x - ct) = -\frac{t - \frac{1}{c}x}{\sqrt{(\frac{1}{c})^2 - 1}}\,. \tag{2.10}$$

See also equation (2.17) below.

The equivalent problem for the magnon was solved in [53], by a method involving finding a scattering solution and analytically continuing it. Like the background solution, the modes can be read off by interchanging x and t.

- First, there is one massless solution (i.e. $\omega^2 = k^2$):

$$\begin{aligned}\delta_r \vec{X} &= e^{ikv-i|k|u}\,\vec{n}\left(k + |k|\cos\frac{p}{2}\right)\operatorname{sech} v \tanh v\,, \\ \delta_r X^1 + i\delta_r X^2 &= -i\,e^{ikv-i|k|u}e^{ix}\left(k - |k|\sinh v \sinh\left(v + i\frac{p}{2}\right)\right)\operatorname{sech}^2 v\,, \\ \delta_r X^1 - i\delta_r X^2 &= i\,e^{ikv-i|k|u}e^{-ix}\left(k - |k|\sinh v \sinh\left(v - i\frac{p}{2}\right)\right)\operatorname{sech}^2 v\,.\end{aligned} \tag{2.11}$$

  This solution we drop on the grounds that it is pure gauge: at any given point (x,t), it is just a linear combination of the zero modes δ_v and δ_x, which we showed to be just reparametrisations.[5] The required combination is

$$\delta_r X^i = e^{ikv-i|k|u}\left(-\left(k + k\cos\frac{p}{2}\right)\delta_v X^i + |k|\,\delta_x X^i\right).$$

- Second, there are three orthogonal fluctuations, in directions $\vec{m}$ with $\vec{m}\cdot\vec{n} = 0$:

$$\begin{aligned}\delta_\perp \vec{X} &= e^{ikv-i\omega u}\,\vec{m}\,(k + i\tanh v)\,, \\ \delta_\perp X^1 &= \delta X^2_\perp = 0,\end{aligned} \tag{2.12}$$

  and one parallel fluctuation, along the spike's orientation $\vec{n}$:

$$\begin{aligned}\delta_{||}\vec{X} &= e^{ikv-i\omega u}\,\vec{n}\left(k + i\tanh v - \left(k + \omega\cos\frac{p}{2}\right)\operatorname{sech}^2 v\right)\,, \\ \delta_{||}X^1 + i\delta_{||}X^2 &= -i\,e^{ikv-i\omega u}e^{ix}\left(k\sinh v + \omega\sinh\left(v + i\frac{p}{2}\right) + i\cosh v\right)\operatorname{sech}^2 v\,, \\ \delta_{||}X^1 - i\delta_{||}X^2 &= i\,e^{ikv-i\omega u}e^{-ix}\left(k\sinh v + \omega\sinh\left(v - i\frac{p}{2}\right) + i\cosh v\right)\operatorname{sech}^2 v\,.\end{aligned} \tag{2.13}$$

  These all have $\omega^2 = k^2 + 1$, the dispersion relation for a massive particle $m^2 = 1$.

Despite appearing massive in u, v, these modes nevertheless represent an instability with respect to physical time. Re-write the modes in the original co-ordinates x, t, by defining the K, W as follows:

$$\delta X^j = e^{ikv-i\omega u}\,f^j(v) = e^{iKx-iWt}\,f^j(\gamma(t-cx)). \tag{2.14}$$

Then $W^2 = K^2 - 1$: written in x, t, these modes are tachyonic (with $m^2 = -1$). In our gauge, $t = X^0$ is the target-space's time coordinate. Since we have no reason to exclude $|K| < 1$, we have modes with imaginary W, which are exponentially growing or dying in time, rather than oscillating.

[5] Note that the breaking of translational symmetry on the worldsheet (discussed in section 2.3) affects only the zero modes.

2.5 Modes in AdS directions

The solutions above are in the $\mathbb{R} \times S^5$ subspace of $AdS_5 \times S^5$. There are no zero modes in the AdS directions, as the centre is a special place, but there are non-zero modes. These are simply the modes of a point particle about the centre of Anti-de Sitter space, identical to the giant magnon case.

Write the AdS_5 part of the metric as

$$ds^2_{\text{AdS}} = -\left(\frac{1+\eta^2/4}{1-\eta^2/4}\right)^2 d\tau^2 + \frac{1}{(1-\eta^2/4)^2} d\eta_k d\eta_k \,,$$

where $k = 1, 2, 3, 4$. In these co-ordinates the modes are simply

$$\eta_k(x,t) = e^{iKx - iWt} f_k(K)$$

with $W^2 = K^2 + 1$. (The infinitely wound hoop has identical AdS modes. We write the Lagrangian for perturbations of this in section 4.1 and all the modes in appendix B.2.)

2.6 The Pohlmeyer map

The theory of classical strings moving on $\mathbb{R} \times S^2$ is equivalent to the sine-gordon model. The mapping goes as follows: if $X^\mu(x,t)$ is a conformal-gauge string solution with $X^0 = t$, then the field $\alpha(x,t)$ defined by [60]

$$\cos\alpha = -\partial_t X^i \partial_t X^i + \partial_x X^i \partial_x X^i \tag{2.15}$$

obeys the sine-gordon equation

$$-\partial_t\partial_t\alpha + \partial_x\partial_x\alpha = \sin\alpha \,.$$

This is the equation of motion for a field with Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_t\alpha)^2 + \frac{1}{2}(\partial_x\alpha)^2 + U(\alpha)$$

(and thus Hamiltonian $\mathcal{H} = \frac{1}{2}(\partial_t\alpha)^2 + \frac{1}{2}(\partial_x\alpha)^2 + U(\alpha)$, in our sign convention) using potential

$$U(\alpha) = 1 - \cos\alpha = 2\sin^2\left(\frac{\alpha}{2}\right) \,.$$

The giant magnon (2.3) is mapped to the simple kink [6]

$$\alpha = 4\arctan\left(e^{-\gamma(x-ct)}\right)$$

connecting $\alpha = 0$ and $\alpha = 2\pi$ at $x = \pm\infty$. The point particle is mapped to the vacuum $\alpha = 0$, and the constant in $U(\alpha)$ was chosen to make its energy (and thus the energy density away from the kink) zero. Then the kink has energy

$$E_{s.g} = 8\gamma = \frac{8}{\sin(p/2)} \,. \tag{2.16}$$

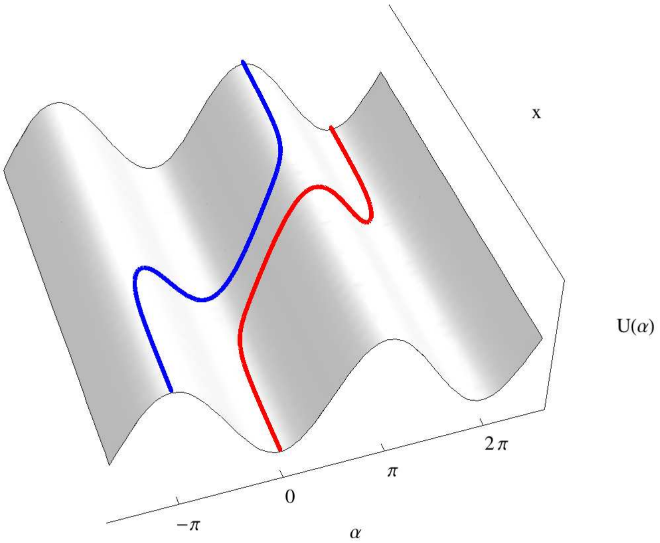

Figure 3: Under the Pohlmeyer map, the magnon is sent to the ordinary kink (in red) while the single spike is mapped to an unstable solution connecting the hilltops (in blue). The sine-gordon field α is plotted left-to-right, x into the page, and $U(\alpha)$ vertically.

The velocity c can be changed by boosting the kink, and the energy $E_{\text{s.g.}}$ changes as one would expect for a relativistic object of rest mass 8.[6] But this energy, from the sine-gordon model's Hamiltonian, is inverse to the spin-chain energy constructed out of target space charges $\Delta - J = \frac{\sqrt{\lambda}}{2}\sin(p/2)$. This mismatch leads to the following difference: while the time-delay of scattering giant magnons (on the string worldsheet) or kinks (in sine-gordon theory) is the same, the resulting phase shift of the wave-functions is different. [6] That is because these two theories are only identical at the classical level. [61]

The single spike (2.5) is mapped instead to an unstable kink. From the map (2.15) it is clear that the effect of the $x \leftrightarrow t$ interchange is to shift the field by π:

$$\alpha(x,t) = \alpha_{\text{magnon}}(t,x) - \pi = 4\arctan\left(e^{-\gamma(t-cx)}\right) - \pi\,.$$

This solution connects two adjacent maxima of $U(\alpha)$, rather than two minima: $\alpha = \pm\pi$ at $x = \pm\infty$. Both cases are drawn in figure 3. If we choose the constant in $U(\alpha)$ to place these maxima at zero

$$U(\alpha) = -1 - \cos\alpha = -2\sin^2\left(\frac{\alpha+\pi}{2}\right),$$

then this unstable kink solution has energy

$$E_{\text{s.g.}} = 8\,c\gamma = \frac{8\cos(p/2)}{\sin(p/2)} = \frac{8}{\sqrt{(\frac{1}{c})^2 - 1}}. \tag{2.17}$$

[6]However, giant magnons of different c are not related by worldsheet boosts (which are just reparametrisations) since $X^0 = t$ is held fixed.

3. Fermionic sector

To check for supersymmetry, we now calculate the fermionic fluctuations of this solution. We find that these are all massless, while 2D supersymmetry would require them to have the same masses as the bosonic modes above. Also, there are twice as many fermionic as bosonic modes, while supersymmetry needs equally many.

The calculation follows what was done for the giant magnon by Minahan [57] (zero modes) and Papathanasiou and Spradlin [53] (non-zero modes).

3.1 Setup

We follow the notation from [57] as much as possible, except for the worldsheet co-ordinates: we use (x, t) and boost by c to (u, v) (instead of (σ, t) and boost by v to (x, ξ)). Indices $a, b = 0, 1$ are worldsheet directions, μ, ν curved spacetime, A, B, C flat spacetime, and $I, J = 1, 2$ number fields.

The unperturbed solution (2.5) lives in $\mathbb{R} \times S^2$, for which we now use co-ordinates t and the usual angles θ and ϕ. This part of the metric is then

$$g_{\mu\nu} = E_\mu^A E_\nu^B \eta_{AB} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & \sin^2\theta \end{bmatrix} \quad \text{for } \nu = \begin{matrix} t \\ \theta \\ \phi \end{matrix}, \tag{3.1}$$

so the vielbein's components are $E_t^t = E_\theta^\theta = 1$ and $E_\phi^\phi = \sin\theta$. (We are using labels t, θ, ϕ for both curved and flat indices.) The single spike (2.5) in these co-ordinates is

$$\begin{aligned} X^0 &= t\,, \\ X^\theta &= \theta = \arccos\left(\frac{1}{\gamma\cosh v}\right), \qquad\qquad \text{i.e. } \cos\theta = \sqrt{1-c^2}\,\mathrm{sech}\, v\,, \\ X^\phi &= \phi = x + \arctan\left(\frac{\tanh v}{c\gamma}\right), \end{aligned}$$

where u, v, γ are still given by (2.4).

The fermionic fluctuations are two Majorana-Weyl fields Θ^I, with action given by Metsaev and Tseytlin [62][7]

$$S = 2\frac{\sqrt{\lambda}}{4\pi}\int dtdx\,\mathcal{L}_F \qquad \text{where} \qquad \mathcal{L}_F = i(\eta^{ab}\delta^{IJ} + \epsilon^{ab}\eta^{IJ})\,\overline{\Theta}^I \rho_a D_b \theta^J\,.$$

The covariant derivative is defined as

$$D_a\Theta^I = \left(\partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}\right)\delta^{IJ}\Theta^J - \frac{i}{2}\Gamma_\star\rho_a\epsilon^{IJ}\Theta^J$$

[7] Note that we use ϵ and η with different kinds of indices: $\epsilon^{ab=01} = 1 = \epsilon^{AB=12}$, and $\eta^{ab=00} = -1 = \eta^{IJ=11}$. Our gamma-matrices are in the all imaginary basis: $\Gamma_{A\neq 0}$ are Hermitian and Γ_0 is anti-Hermitian. $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]}$, thus $\Gamma_{\phi\theta} = \Gamma_\phi\Gamma_\theta$.

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where $\Gamma_\star = i\Gamma_{01234} = i\Gamma_{[0}\Gamma_1\Gamma_2\Gamma_3\Gamma_{4]}$ (these are the AdS directions) has $\Gamma_\star^2 = 1$. This action leads to the following equations of motion:

$$(\rho_0 - \rho_1)\,(D_0 + D_1)\,\Theta^1 = 0\,,$$
$$(\rho_0 + \rho_1)\,(D_0 - D_1)\,\Theta^2 = 0\,.$$

The projections of the gamma matrices $\rho_a = \Gamma_A E^A_\mu \partial_a X^\mu$ and the spin connection $\omega_a^{AB} = \omega_\mu^{AB}\partial_a X^\mu$ are:[8]

$$\begin{aligned}
\rho_0 &= \Gamma_0 + c\gamma^2\frac{\cos^2\theta}{\sin\theta}\Gamma_\phi + \gamma^2\frac{\cos\theta}{\sin\theta}\sqrt{\sin^2\theta - c^2}\Gamma_\theta && = \Gamma_0 + r(\theta)\Gamma_\phi + s(\theta)\Gamma_\theta\,,\\
\rho_1 &= \gamma^2\frac{\sin^2\theta - c^2}{\sin\theta}\Gamma_\phi - c\gamma^2\frac{\cos\theta}{\sin\theta}\sqrt{\sin^2\theta - c^2}\Gamma_\theta && = p(\theta)\Gamma_\phi + q(\theta)\Gamma_\theta\,,\\
\omega_0 &= -\omega_0^{\phi\theta} = -c\gamma^2\frac{\cos^3\theta}{\sin^2\theta}\,,\\
\omega_1 &= -\omega_1^{\phi\theta} = -\gamma^2\frac{\cos\theta}{\sin^2\theta}(\sin^2\theta - c^2)\,.
\end{aligned}$$

The first step is to replace $\partial_0 = \partial_t$ and $\partial_1 = \partial_x$ with the boosted derivatives $\partial_u = \gamma(\partial_1 + c\partial_0)$ and $\partial_v = \gamma(\partial_0 + c\partial_1)$, thus

$$\partial_0 \pm \partial_1 = (1 \mp c)\gamma\left\{\partial_u \pm \partial_v\right\}\,.$$

Following this pattern, define G and $\tilde G$ as follows:

$$\begin{aligned}
\omega_0^{\phi\theta} + \omega_1^{\phi\theta} &= (1-c)\gamma\frac{1}{2}G, &\quad \text{where}\quad G &= \gamma\frac{\cos\theta}{\sin^2\theta}(c + \sin^2\theta),\\
\omega_0^{\phi\theta} - \omega_1^{\phi\theta} &= (1+c)\gamma\frac{1}{2}\tilde G, & \tilde G &= \gamma\frac{\cos\theta}{\sin^2\theta}(c - \sin^2\theta).
\end{aligned}$$

The equations of motion can then be written as

$$\begin{aligned}
(\rho_0 - \rho_1)\left[(1-c)\gamma\left\{\partial_v + \partial_u + \frac{1}{2}G\Gamma_{\phi\theta}\right\}\Theta^1 - \frac{i}{2}\Gamma_\star(\rho_0 + \rho_1)\Theta^2\right] &= 0,\\
(\rho_0 + \rho_1)\left[(1+c)\gamma\left\{\partial_v - \partial_u + \frac{1}{2}\tilde G\Gamma_{\phi\theta}\right\}\Theta^2 + \frac{i}{2}\Gamma_\star(\rho_0 - \rho_1)\Theta^1\right] &= 0.
\end{aligned} \tag{3.2}$$

It is useful to define operators

$$\mathcal{D}_v = \partial_v + \frac{1}{2}G\Gamma_{\phi\theta}\,, \qquad \tilde{\mathcal{D}}_v = \partial_v + \frac{1}{2}\tilde G\Gamma_{\phi\theta}\,,$$

so that the curly brackets in the equations of motion (3.2) are these operators plus or minus the time derivative ∂_u.

[8] As functions of θ, these are simply related to their cousins in the giant magnon case: $\rho_0 = \rho_1^{\mathrm{magnon}} + \Gamma_0$, $\rho_1 = \rho_0^{\mathrm{magnon}} - \Gamma_0$, $\omega_0 = \omega_1^{\mathrm{magnon}}$ and $\omega_1 = \omega_0^{\mathrm{magnon}}$. We took our conventions for the spin connection from [63]. Functions p, q, r, s are useful in what follows.

We now want to write the equations of motion in terms of kappa-symmetry fixed fields, [64, 57, 53] which we define as

$$\Psi^1 = -i(\rho_0 - \rho_1)\Theta^1\,, \qquad (3.3)$$
$$\Psi^2 = i(\rho_0 + \rho_1)\Theta^2\,.$$

Note that Γ_{11} anti-commutes with $i(\rho_0 \pm \rho_1)$, and that these operators are real. Thus Θ^I is Majorana-Weyl exactly when Ψ^I is, so we will impose the conditions on Ψ^I.

To write the equations of motion in terms of these symmetry-fixed fields, we will need several identities, identical in form to those in the giant magnon case [57].[9] The following two operators are nilpotent:

$$(\rho_0 \pm \rho_1)^2 = 0$$

(thus $(\rho_0 - \rho_1)\Psi^1 = 0$ and $(\rho_0 + \rho_1)\Psi^2 = 0$) and can be shown to commute with the curly derivatives

$$[\mathcal{D}_v, (\rho_0 - \rho_1)] = 0\,, \qquad \left[\tilde{\mathcal{D}}_v, (\rho_0 + \rho_1)\right] = 0\,.$$

(They trivially commute with ∂_u too.) Also important is the dagger of ρ_0:

$$\overline{\rho}_0 \equiv \Gamma_\star \rho_0 \Gamma_\star = -\rho_0^\dagger = \Gamma_0 - r\Gamma_\phi - s\Gamma_\theta,$$

which allows us to write two more nilpotent operators

$$(\overline{\rho}_0 \pm \rho_1)^2 = 0$$

as well as a non-singular operator $(\overline{\rho}_0 - \rho_0) = -2r\Gamma_\phi - 2s\Gamma_\theta$, whose square is proportional to the unit matrix:

$$(\overline{\rho}_0 - \rho_0)^2 = 4\gamma^2 \cos^2\theta.$$

Returning to the equations of motion (3.2), we can now pull the operators $(\rho_0 \pm \rho_1)$ to the right, using the identities above, until they act on the Θ^I to give Ψ^I. We obtain:

$$(1-c)\gamma\left\{\mathcal{D}_v + \partial_u\right\}\Psi^1 + \frac{i}{2}\Gamma_\star(\overline{\rho}_0 + \rho_0)\Psi^2 = 0\,, \qquad (3.4)$$
$$(1+c)\gamma\left\{\tilde{\mathcal{D}}_v - \partial_u\right\}\Psi^2 - \frac{i}{2}\Gamma_\star(\overline{\rho}_0 - \rho_0)\Psi^1 = 0\,.$$

3.2 Non-zero modes

Begin by solving the first of equations (3.4) for Ψ^2 :

$$\Psi^2 = \frac{(\overline{\rho}_0 - \rho_0)}{4\gamma^2\cos^2\theta}\Gamma_\star \frac{2}{i}(1-c)\gamma\left\{\mathcal{D}_v + \partial_u\right\}\Psi^1. \qquad (3.5)$$

[9] To derive these identities, write relations such as $(\rho_0 \pm \rho_1)^2 = -1 + (r \pm p)^2 + (s \pm q)^2$ and $(\overline{\rho}_0 \pm \rho_1)^2 = -1 + (-r \pm p)^2 + (-s \pm q)^2$.

We can then eliminate Ψ^2 from the other equation to obtain a second-order equation for Ψ^1 alone:

$$\left\{\tilde{\mathcal{D}}_v - \partial_u\right\} \frac{(\overline{\rho}_0 - \rho_0)}{\gamma^2 \cos^2\theta} \left\{\mathcal{D}_v + \partial_u\right\} \Psi^1 + (\overline{\rho}_0 - \rho_0)\Psi^1 = 0.$$

Using the identity

$$\left\{\tilde{\mathcal{D}}_v - \partial_u\right\} \frac{(\overline{\rho}_0 - \rho_0)}{\cos\theta} = \frac{(\overline{\rho}_0 - \rho_0)}{\cos\theta} \left\{\mathcal{D}_v - \partial_u\right\}$$

and pulling the $(\rho_0 - \rho_1)$ from Ψ^1's definition through, this becomes

$$(\rho_0 - \rho_1)\left(\frac{1}{\gamma\cos\theta}\left\{\mathcal{D}_v - \partial_u\right\}\frac{1}{\gamma\cos\theta}\left\{\mathcal{D}_v + \partial_u\right\} + 1\right)\Theta^1 = 0\,, \tag{3.6}$$

analogous to equation (3.7) of [53]. We can solve this equation by a similar method to the one used there: we temporarily drop the kappa-symmetry projection $(\rho_0 - \rho_1)$, and solve the remainder of the equation for Θ^1. At the end we will apply the projection to recover Ψ^1, and from that will find Ψ^2 using (3.5).

To find Θ^1, we split this second-order equation (3.6) into two first-order equations, defining some intermediate field $\tilde{\Theta}$:

$$\begin{bmatrix} \mathcal{D}_v + \partial_u & -i\,\mathrm{sech}\, v \\ -i\,\mathrm{sech}\, v & \mathcal{D}_v - \partial_u \end{bmatrix} \begin{pmatrix} \Theta^1 \\ \tilde{\Theta} \end{pmatrix} = 0\,.$$

We expand the spinor in a Fourier series for u:

$$\begin{pmatrix} \Theta^1\,(u,v) \\ \tilde{\Theta}\,(u,v) \end{pmatrix} = e^{-i\omega u}\vec{\Theta}(v,\omega)\,, \tag{3.7}$$

and also into a sum of eigenspinors of $\Gamma_{\phi\theta}$: $\vec{\Theta} = \vec{\Theta}_+ + \vec{\Theta}_-$ with

$$(1_{2\times 2}\otimes\Gamma_{\phi\theta})\;\vec{\Theta}_\pm = \pm i\,\vec{\Theta}_\pm.$$

Then the coupled linear equations can be written as

$$(\partial_v - V_\pm)\,\vec{\Theta}_\pm = 0, \qquad \text{with} \quad V_\pm = \begin{bmatrix} i\left(\omega \mp \frac{G}{2}\right) & i\,\mathrm{sech}\, v \\ i\,\mathrm{sech}\, v & -i\left(\omega \pm \frac{G}{2}\right) \end{bmatrix}.$$

We now proceed to diagonalize the system of equations, by a change of basis.

Diagonalisation. Define $\vec{\Theta}'_\pm = S\vec{\Theta}_\pm$, which obeys

$$\partial_v\vec{\Theta}'_\pm = (\partial_v S + SV_\pm)S^{-1}\vec{\Theta}'_\pm = H_\pm\vec{\Theta}'_\pm \tag{3.8}$$

(defining $H_\pm$). We want to choose S to make $H_\pm$ diagonal. If we write

$$S = \begin{bmatrix} a(v) & b(v) \\ c(v) & d(v) \end{bmatrix},$$

then setting the off-diagonal elements of $H_\pm$ to zero reads

$$\begin{aligned} 0 &= i(a^2 - b^2)\,\mathrm{sech}\, v - 2i\omega ab - ba' + ab', \\ 0 &= i(d^2 - c^2)\,\mathrm{sech}\, v + 2i\omega cd - cd' + dc'. \end{aligned}$$

One obtains the same equations for both H_+ and H_-, which means S diagonalizes both simultaneously. Because we have only two equations and four parameters, we choose two additional relations among these four entries[10]

$$\begin{aligned} a' &= -ib\,\mathrm{sech}(v)\,, \\ c' &= -id\,\mathrm{sech}(v)\,, \end{aligned} \tag{3.9}$$

leading to a second-order equation for a (and c obeying the same equation):

$$-a'' - \tanh v\, a' + 2i\omega a' - \mathrm{sech}^2\, v\, a = 0. \tag{3.10}$$

It also leads to this simple form for $H_\pm$:

$$H_\pm = i\left(\omega \mp \frac{G}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.11}$$

The two solutions to (3.10) are

$$\begin{aligned} a_1\,(v) &= \frac{2i\omega}{1+4\omega^2} + \frac{\tanh v}{1+4\omega^2}\,, \\ a_2\,(v) &= e^{2i\omega v}\,\mathrm{sech}\, v\,, \end{aligned}$$

and $a(v)$ and $c(v)$ are (different) linear combinations of these. The other functions $b(v)$ and $d(v)$ are then fixed by (3.9). We can write the general solution for S as

$$S = S_0 \begin{bmatrix} a_1(v) & b_1(v) \\ a_2(v) & b_2(v) \end{bmatrix}$$

where S_0 is a non-singular constant matrix, and

$$\begin{aligned} b_1\,(v) &= i\frac{\mathrm{sech}\, v}{1+4\omega^2}\,, \\ b_2\,(v) &= i\,(2i\omega - \tanh v)\, e^{2i\omega v}\,. \end{aligned}$$

The determinant of this change of basis is $\det S = -ie^{2i\omega v}\,\det S_0$, different from zero, as expected.

[10]These extra relations can be imposed by multiplying S by a non-singular diagonal matrix, which is always allowed as it does not change the equations of motion (3.8).

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Solving. We can now solve the diagonalized system (3.8), using $H_\pm$ from (3.11). The equations become simply

$$\left(\partial_v - i\left(\omega \mp \frac{G}{2}\right)\right) f(v) = 0\,.$$

A very similar equation occurs in the magnon case [57] (and also for the zero modes in appendix A). It has solution $f(v) = e^{\pm i\chi} e^{i\omega v}$, where

$$e^{i\chi} = \left(\frac{\sinh v + ic}{\sinh v - ic}\right)^{1/4} \sqrt{\tanh v + i\,\mathrm{sech}\, v}.$$

Thus $\vec{\Theta}'_\pm$ will be given by this phase times a spinor:

$$\vec{\Theta}'_\pm = e^{\pm i\chi} e^{i\omega v} \vec{U}_\pm\,,$$

where $\vec{U}_\pm$ is any eigenspinor of $(1 \otimes \Gamma_{\phi\theta})$ with eigenvalues $\pm i$. It remains to rotate back to unprimed $\vec{\Theta}_\pm$, which is

$$\vec{\Theta}_\pm = S^{-1} \vec{\Theta}_\pm = e^{\pm i\chi} e^{i\omega v} S^{-1} \vec{U}_\pm.$$

We can now absorb the constant matrix S_0^{-1} into the arbitrary spinor $\vec{U}_\pm$, which we do by writing

$$S_0^{-1} \vec{U}_\pm = \frac{1}{\sqrt{1-c}} \begin{pmatrix} U_\pm \\ \tilde{U}_\pm \end{pmatrix}\,,$$

introducing $U_\pm$ and $\tilde{U}_\pm$, and slipping in the $\sqrt{1-c}$ for later convenience.

Recall from (3.7) that our original spinor Θ^1 is the first component of $e^{-i\omega u}\left(\vec{\Theta}_+ + \vec{\Theta}_-\right)$. We can now read it off, obtaining[11]

$$\begin{aligned}
\Theta^1(u,v) &= \frac{1}{\sqrt{1-c}} e^{-i\omega u} \sum_\pm \frac{e^{i\omega v \pm i\chi}}{-ie^{2i\omega v}} \left[b_2(v)\, U_\pm - b_1(v)\, \tilde{U}_\pm\right] \\
&= \frac{-1}{\sqrt{1-c}} e^{-i\omega u} \sum_\pm e^{\pm i\chi} \left[e^{-i\omega v} \frac{\mathrm{sech}\, v}{1+4\omega^2} U_\pm + (\tanh v - 2i\omega)\, e^{i\omega v} \tilde{U}_\pm\right].
\end{aligned}$$

To get the symmetry-fixed field $\Psi^1 = -i(\rho_0 - \rho_1)\Theta^1$ it is useful to use the identity $e^{\pm 2i\chi} = (p-r) \mp i(q-s)$. We find the following positive-frequency solution:

$$\begin{aligned}
\Psi^1_p &= \frac{i\, e^{-i\omega u}}{\sqrt{1-c}} \sum_\pm \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[e^{-i\omega v} \frac{\mathrm{sech}\, v}{1+4\omega^2} U_\pm + (\tanh v - 2i\omega)\, e^{i\omega v} \tilde{U}_\pm\right] \\
&= \frac{i}{\sqrt{1-c}} \sum_\pm \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[e^{i\alpha} \frac{\mathrm{sech}\, v}{1+4\omega^2} U_\pm + \sqrt{\tanh^2 v + 4\omega^2}\, e^{i\beta} \tilde{U}_\pm\right] \qquad (3.12)
\end{aligned}$$

where the phases α and β are defined by

$$\begin{aligned}
e^{i\alpha} &= e^{-i\omega(u+v)}, \\
e^{i\beta} &= e^{-i\omega(u-v)} e^{-i\arctan(2\omega \coth v)}.
\end{aligned}$$

[11] The second entry will be given by

$$\tilde{\Theta}(u,v) = -ie^{-i\omega u} e^{\pm i\chi} \left[e^{i\omega v}\, \mathrm{sech}\, v\, U_\pm - \frac{(\tanh v + 2i\omega)}{1+4\omega^2} e^{-i\omega v} \tilde{U}_\pm\right].$$

This is useful for finding Ψ^2 later.

Majorana condition. It remains to impose the Majorana condition on the spinors, that is, Ψ^I should be real $\Psi^{I*} = \Psi^I$. To do so, we have to consider a superposition of positive and negative frequencies ω. We thus write

$$\begin{aligned}\Psi^1 &= 2\,\mathrm{Re}\,\Psi^1_p = \Psi^1_p + \Psi^{1*}_p \\ &= \frac{i}{\sqrt{1-c}} \sum_{\pm} \left(e^{\pm i\chi}\Gamma_0 - e^{\mp i\chi}\Gamma_\phi\right) \left[\frac{\mathrm{sech}\, v}{1+4\omega^2}\left(e^{i\alpha}U_\pm + e^{-i\alpha}U^*_\mp\right)\right. \\ &\qquad\qquad \left. + \sqrt{\tanh^2 v + 4\omega^2}\left(e^{i\beta}\tilde{U}_\pm + e^{-i\beta}\tilde{U}^*_\mp\right)\right].\end{aligned}$$

Note that $U^*_\mp$ is an eigenspinor of $\Gamma_{\phi\theta}$ of eigenvalue $\pm i$. (The Γ matrices are imaginary, thus $\Gamma_{\phi\theta}$ is real.)

Combine the four $\pm$ eigenspinors into two spinors $U = U_+ + U_-$ and $\tilde{U} = \tilde{U}_+ + \tilde{U}_-$. (We can reverse this with projection operators $U_\pm = \frac{i \pm \Gamma_{\phi\theta}}{2i} U$, and similarly for the others.). Then we can write

$$\begin{aligned}\Psi^1 &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\ &\qquad \times \left\{\frac{\mathrm{sech}\, v}{1+4\omega^2}\,\mathrm{Re}(e^{i\alpha}U) + \sqrt{\tanh^2 v + 4\omega^2}\,\mathrm{Re}(e^{i\beta}\tilde{U})\right\} \\ &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\ &\qquad \times \left\{\frac{\mathrm{sech}\, v}{1+4\omega^2}\left(\cos\alpha\, U_0 + \sin\alpha\,\Gamma_{\phi\theta}U_1\right)\right. \\ &\qquad\qquad \left. + \sqrt{\tanh^2 v + 4\omega^2}\left(\cos\beta\,\tilde{U}_0 + \sin\beta\,\Gamma_{\phi\theta}\tilde{U}_1\right)\right\},\end{aligned} \tag{3.13}$$

where the new spinors are

$$\begin{aligned} U_0 &= 2\,\mathrm{Re}\left(U_+ + U_-\right), & \tilde{U}_0 &= 2\,\mathrm{Re}\left(\tilde{U}_+ + \tilde{U}_-\right), \\ U_1 &= 2\,\mathrm{Re}\left(U_+ - U_-\right), & \tilde{U}_1 &= 2\,\mathrm{Re}\left(\tilde{U}_+ - \tilde{U}_-\right), \end{aligned} \tag{3.14}$$

thus $U_0 = 2\,\mathrm{Re}\,U$, but $U_1 = 2\Gamma_{\phi\theta}\,\mathrm{Im}\,U$ (and similarly with tildes).

We can now find Ψ^2 from Ψ^1 using (3.5). The final, Majorana, field is

$$\begin{aligned}\Psi^2 &= \frac{1}{\sqrt{1+c}}\Gamma_*\Gamma_\theta\left[\Gamma_0\left(\cos\tilde{\chi} + \Gamma_{\phi\theta}\sin\tilde{\chi}\right) - \Gamma_\phi\left(\cos\tilde{\chi} - \Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right] \\ &\qquad \times \left\{\mathrm{sech}\, v\left(\cos\tilde{\alpha}\,\tilde{U}_0 + \sin\tilde{\alpha}\,\Gamma_{\phi\theta}\tilde{U}_1\right)\right. \\ &\qquad\qquad \left. - \frac{\sqrt{\tanh^2 v + 4\omega^2}}{1+4\omega^2}\left(\cos\tilde{\beta}\,U_0 + \sin\tilde{\beta}\,\Gamma_{\phi\theta}U_1\right)\right\},\end{aligned} \tag{3.15}$$

where the new phases are

$$\begin{aligned} e^{i\tilde{\chi}} &= \sqrt{\frac{\sinh v - ic}{\sinh v + ic}}\,e^{i\chi} = \left(\frac{\sinh v - ic}{\sinh v + ic}\right)^{1/4}\sqrt{\tanh v + i\,\mathrm{sech}\, v}, \\ e^{i\tilde{\alpha}} &= e^{-i\omega(u-v)}, \\ e^{i\tilde{\beta}} &= e^{-i\omega(u+v)}e^{i\arctan(2\omega\coth v)}. \end{aligned}$$

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3.3 Mass and counting

The phases appearing in Ψ^1, far from the spike ($|v| \gg 1$), are $i\alpha = -i\omega u - i\omega v$ and $i\beta = -i\omega u + i\omega v$. This means that the fermionic modes are massless, $\omega^2 = k^2$. But the bosonic modes we found in section 2 are not massless, so there can be no supersymmetry.

How many fermionic modes are there? There are four spinors $U_\pm$ and $\tilde{U}_\pm$, which are $\Gamma_{\phi\theta}$ eigenspinors, so have 16 complex components each. They must also be Γ_{11} eigenspinors, for Ψ to be Weyl, cutting the number by half. And then we found in (3.14) that the Majorana spinor depends only on the real part of each, cutting it in half again. This leaves 16 complex degrees of freedom, which is twice the number for the giant magnon. [53]

The bosonic modes were obtained simply by switching $x \leftrightarrow t$ in the magnon expressions. So their number is unchanged from the magnon case: there are 8 non-zero modes (4 on the sphere and 4 in AdS). The fact that there are two fermionic modes for each bosonic modes is a second piece of evidence against supersymmetry.

There are also twice as many fermionic zero modes (8 complex) as bosonic zero modes (4, as for the magnon). Because the non-zero modes are massless, $\omega = 0$ is part of the continuum, and expressions for the zero modes can be found by simply setting $\omega = 0$ in (3.13) and (3.15) above (which sets $\alpha = \beta = 0$). But the counting is more delicate, the zero modes appear to have the same dependence on $\operatorname{Re} U_\pm$ and $\operatorname{Re} \tilde{U}_\pm$ as the non-zero modes, suggesting that there are also 16 of them. However, the same argument as used for the magnon case [57] kills half of these, leaving 8. We perform the exact analogue of [57]'s calculation of the magnon zero modes in appendix A.

4. Quantum corrections

4.1 Corrections to what?

Having found the modes, it would be natural to use them to compute a first quantum correction, i.e. to perform 'semi-classical quantisation'. For the giant magnon, this means finding quantum corrections to $\Delta - J$. The origin of this is as follows:

Frolov and Tseytlin [39] consider the 'vacuum' of the large-J sector, the point particle orbiting the sphere, which has $\Delta = J$. They add small perturbations to this, and show that $\Delta - J$ is (at leading order in $1/\sqrt{\lambda}$) the Hamiltonian of a 1+1-dimensional theory. The perpendicular fluctuations in both the sphere and AdS are non-interacting massive fields of this theory. So far this is classical. The semi-classical correction is to treat each mode of these fields as a harmonic oscillator, and their zero-point energies '$\frac{1}{2}\hbar\omega$' are corrections to $\Delta - J$. The magnon is interpreted as a 'giant perturbation' of this vacuum, tall enough to see the curvature of spacetime. (And, it turns out, of high enough momentum to see that the 1+1-dimensional theory is a spin chain, with periodic dispersion relation.)

Here we repeat their calculation, for the 'vacuum of the large-winding sector': the infinitely wound hoop. We find as Hamiltonian $\Delta - \Phi$, with the winding charge Φ replacing the angular momentum J. The single spike is similarly a 'giant perturbation' of this vacuum.

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Recall from section 2.1 that the flat-space versions of these two classes of spiky strings are related by T-duality, which famously exchanges winding and momentum around a compact direction. Clearly this change in the Hamiltonian is somehow a consequence of this duality. But notice that the compact direction here is part of a sphere, and that the radius of this sphere is unchanged.

Finding the hamiltonian. Write the metric in the form[12]

$$\begin{aligned} ds^2_{\rm AdS} &= -\left(\frac{1+\eta^2/4}{1-\eta^2/4}\right)^2 d\tau^2 + \frac{1}{(1-\eta^2/4)^2} d\eta_k d\eta_k \qquad\qquad k=1,2,3,4 \\ ds^2_{\rm S} &= d\theta_1^2 + \cos^2\theta_1\left(d\theta_2^2 + \cos^2\theta_2\left(d\theta_2^2 + \cos^2\theta_2\left(d\theta_3^2 + \cos^2\theta_3\left(d\theta_4^2+\cos^2\theta_4 d\phi^2\right)\right)\right)\right) . \end{aligned}$$

The action (in conformal gauge) is

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int dx dt\, \mathcal{L}_B, \qquad\qquad \mathcal{L}_B = \frac{1}{2}\partial^a X^\mu \partial_a X^\nu G_{\mu\nu} \,. \tag{4.1}$$

We write the perturbed the solution as $X^\mu = X^\mu_{\rm hoop} + \tilde{X}^\mu/\lambda^{1/4}$:

$$\begin{aligned} \tau &= t + \frac{1}{\lambda^{1/4}}\tilde{\tau} & \phi &= x + \frac{1}{\lambda^{1/4}}\tilde{\phi} \\ \eta_k &= \frac{1}{\lambda^{1/4}}\tilde{\eta}_k & \theta_s &= \frac{1}{\lambda^{1/4}}\tilde{\theta}_s, \qquad s=1,2,3,4. \end{aligned} \tag{4.2}$$

Expanding at large λ, the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_B = 1 &+ \frac{1}{\lambda^{1/4}}\left(\partial_0\tilde{\tau} + \partial_1\tilde{\phi}\right) \\ &+ \frac{1}{2\sqrt{\lambda}}\left(-\partial^a\tilde{\tau}\partial_a\tilde{\tau} + \partial^a\tilde{\eta}_k\partial_a\tilde{\eta}_k + \partial^a\tilde{\phi}\partial_a\tilde{\phi} + \partial^a\tilde{\theta}_s\partial_a\tilde{\theta}_s + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) \\ &+ \frac{1}{\lambda^{3/4}}\left((\partial_0\tilde{\tau})\tilde{\eta}_k\tilde{\eta}_k - (\partial_1\tilde{\phi})\tilde{\theta}_s\tilde{\theta}_s\right) + \mathcal{O}\left(\frac{1}{\lambda}\right). \end{aligned} \tag{4.3}$$

In the quadratic piece, $\tilde{\eta}_k$ appears massive and $\tilde{\theta}_s$ tachyonic, matching what we found for the single spike's modes.

The Virasoro constraints are first $\gamma_{00}+\gamma_{11} = 2T_{00} = 0$:

$$\begin{aligned} 0 = &\frac{1}{\lambda^{1/4}}\left(-\partial_0\tilde{\tau} + \partial_1\tilde{\phi}\right) \\ &+ \frac{1}{2\sqrt{\lambda}}\left(-\partial_a\tilde{\tau}\partial_a\tilde{\tau} + \partial_a\tilde{\eta}_k\partial_a\tilde{\eta}_k + \partial_a\tilde{\phi}\partial_a\tilde{\phi} + \partial_a\tilde{\theta}_s\partial_a\tilde{\theta}_s - \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) \\ &+ \mathcal{O}\left(\frac{1}{\lambda^{3/4}}\right), \end{aligned} \tag{4.4}$$

[12]The azimuthal angle ϕ here is the same as used before, in (3.1), but $\theta_4 = \pi/2 - \theta$ is the elevation above the equator. The expansions of the metric components which we need are $G_{\tau\tau} = -1 - \eta^2 + \cdots$ and $G_{\theta\theta} = 1 - \theta^2 + \cdots$.

(writing $\partial_a\partial_a = \partial_0\partial_0 + \partial_1\partial_1$ in a temporary abuse of notation) and second $\gamma_{01} = T_{01} = 0$:

$$0 = \frac{1}{\lambda^{1/4}}\left(-\partial_1\tilde{\tau} + \partial_0\tilde{\phi}\right) + \frac{1}{2\sqrt{\lambda}}\left(-\partial_0\tilde{\tau}\partial_1\tilde{\tau} + \partial_0\tilde{\phi}\partial_1\tilde{\phi} + \partial_0\tilde{\eta}_k\partial_1\tilde{\eta}_k + \partial_0\tilde{\theta}_s\partial_1\tilde{\theta}_s\right) + \mathcal{O}\left(\frac{1}{\lambda^{3/4}}\right).$$

Now we expand the spacetime charges: the energy is the integral of the momentum density Π^0_τ:

$$\begin{aligned}\Delta &= \frac{1}{2\pi}\int dx\, \frac{\partial \mathcal{L}_B}{\partial\, \partial_0\tau} \\ &= \frac{1}{2\pi}\int dx \left(\sqrt{\lambda} + \lambda^{1/4}\partial_0\tilde{\tau} + \tilde{\eta}_k\tilde{\eta}_k + \mathcal{O}\left(\frac{1}{\lambda^{1/4}}\right)\right),\end{aligned}$$

and the winding charge defined in (2.6) is

$$\begin{aligned}\Phi &= \frac{\sqrt{\lambda}}{2\pi}\int dx\, \partial_1\phi \\ &= \frac{1}{2\pi}\int dx \left(\sqrt{\lambda} + \lambda^{1/4}\partial_1\tilde{\phi}\right).\end{aligned}$$

Subtracting these two charges, the two $\sqrt{\lambda}$ terms will cancel, leaving a finite result. The linear terms can then be replaced with quadratic terms using the first Virasoro constraint (4.4). To leading order in $1/\lambda$, we obtain:

$$\begin{aligned}\Delta - \Phi = \frac{1}{4\pi}\int dx\Big[&-(\partial_0\tilde{\tau}\partial_0\tilde{\tau} + \partial_1\tilde{\tau}\partial_1\tilde{\tau}) + \left(\partial_0\tilde{\phi}\partial_0\tilde{\phi} + \partial_1\tilde{\phi}\partial_1\tilde{\phi}\right) \\ &+ (\partial_0\tilde{\eta}_k\partial_0\tilde{\eta}_k + \partial_1\tilde{\eta}_k\partial_1\tilde{\eta}_k) + \left(\partial_0\tilde{\theta}_s\partial_0\tilde{\theta}_s + \partial_1\tilde{\theta}_s\partial_1\tilde{\theta}_s\right) + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\Big]. \qquad (4.5)\end{aligned}$$

This is the analogue of the result in [39]. The fields $\tilde{\tau}$ and $\tilde{\phi}$ correspond to transformations that are pure gauge, so we drop them. We can write $\Delta - \Phi$ in terms of the Hamiltonian one would obtain from only the quadratic part of the Lagrangian $\mathcal{L}_B$ (see appendix B.1), which contains the transverse (physical) modes $\tilde{\eta}_k$ and $\tilde{\theta}_s$ and their conjugate momenta $\tilde{\Pi}_{\tilde{\eta}_k}, \tilde{\Pi}_{\tilde{\theta}_s}$:

$$\begin{aligned}\Delta - \Phi &= \int \frac{dx}{2\pi}\mathcal{H}_{2d}\left(\tilde{\tau}, \tilde{\phi}, \tilde{\eta}_k, \tilde{\theta}_s\right) \\ &= \sqrt{\lambda}\int \frac{dx}{4\pi}\left[\tilde{\Pi}^2_{\tilde{\eta}_k} + \tilde{\Pi}^2_{\tilde{\theta}_s} + \partial_1\tilde{\eta}_k\partial_1\tilde{\eta}_k + \partial_1\tilde{\theta}_s\partial_1\tilde{\theta}_s + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right].\end{aligned}$$

We are left with four massive fields from vibrations in the AdS directions and four tachyonic fields from the sphere directions. Then $\Delta - \Phi$ is the expected quadratic Hamiltonian for these 8 fields. One could perform a similar construction for the fermionic modes obtaining 16 massless fermionic fields. [65, 66, 39]

4.2 First quantum correction

For each of the eight bosonic modes $\tilde{\eta}_k$ and $\tilde{\theta}_s$, we have a quadratic Hamiltonian of the kind

$$H_2 = \int dx\left[\frac{1}{2}\hat{\Pi}^2 + \hat{\phi}\left(-\partial_x^2 + V\right)\hat{\phi}\right].$$

Note that $V = \pm 1$ in our case, depending on whether the mode is massive or tachyonic. We can expand both $\hat{\Pi}$ and $\hat{\phi}$ eigenfunctions ψ_n of the differential operator $\left(-\partial_x^2 + V\right)\psi_n = \omega_n^2 \psi_n$, which we write $\hat{\phi} = \sum \hat{\phi}_n \psi_n$ and $\hat{\Pi} = \sum \hat{\Pi}_n \psi_n$. The Hamiltonian becomes a sum of decoupled harmonic oscillators

$$H_2 = \sum \frac{1}{2}\left(\hat{\Pi}_n^2 + \omega_n^2 \hat{\phi}_n^2\right).$$

By introducing creation and annihilation operators in the usual way, for each oscillator, we find that each of these contributes $\frac{1}{2}\sum \hbar\omega_n$, with[13] $\omega_n = \sqrt{k_n^2 + m^2}$, for some mass m^2 and allowed momenta k_n.

For our solution the bosonic modes of section 2 have $W(K) = \sqrt{K^2 \pm 1}$. Each of the fermionic modes will contribute $-\frac{1}{2}\sum \hbar W_{\text{fermi}}$, where the fermionic modes found in section 3 have $W(K) = K$.

There are two important issues here:

- First, to obtain a finite first quantum correction for any solution, one must always subtract the quantum correction for the corresponding vacuum solution. Both of these are normally UV divergent (and this subtraction is not the only renormalization usually needed). For the single spike, the relevant vacuum is the hoop solution. Note that the hoop has $\Delta - \Phi = 0$ classically, so this subtraction is only needed for the quantum corrections.

- Second, we are interested in studying those modes of the spike which result in its instability. To determine the decay time of this unstable solution, we are only interested in the imaginary part of the energy correction. None of the fermionic modes will contribute to this, as they are massless, nor will the 4 bosonic modes in AdS_5, as they are massive. The only contribution is from the 4 tachyonic modes on the sphere, which have $W(K) = \pm\sqrt{K^2 - 1}$, and here only from those modes with $|K| < 1$. This excludes the UV modes, and in fact no other renormalization will be needed.

Vacuum. The bosonic and fermionic modes for the hoop can be found in appendix B.2. They have the same masses as their counterparts for the single spike, in particular the sphere modes have $W(K) = \pm\sqrt{K^2 - 1}$. To discretise the momentum K, we put the solution in a box $-\frac{L}{2} < x < \frac{L}{2}$ and impose periodic boundary conditions $\delta X\left(-\frac{L}{2}\right) = \delta X\left(\frac{L}{2}\right)$. Then $K_n = \frac{2\pi n}{L}$, with $n \in \mathbb{Z}$, and the contribution of these modes to the vacuum energy is given by

$$\begin{aligned} \Delta E_{\text{hoop}} &= 4\frac{1}{2}\sum_n \sqrt{K_n^2 - 1} \\ &\approx 2\frac{L}{2\pi}\int_{-1}^{1} dK\, \sqrt{K^2 - 1} \quad \text{as } L \to \infty \\ &= \frac{i}{2}L\,. \end{aligned} \tag{4.6}$$

The integration is over $|K| < 1$ because we are looking for just the imaginary part. We do not encounter a UV divergence here.

[13]In the literature, $\nu_n = T\omega_n$ (where T is some large time) is called a stability angle.

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Spike solution. Again we study only the bosonic modes on the sphere with $|K| < 1$. But the discrete momenta K allowed for the spike are not the same as those for the hoop K_n, as the modes have a phase shift at large x compared to the hoop. Looking at the bosonic sphere modes given in (2.12) and (2.13), far away from the spike ($|v| \gg 1$) we have

$$\begin{aligned} \delta_\perp \vec{X}\,(x) &= e^{iKx - i\sqrt{K^2-1}t}\,\vec{m}\left[\gamma\,(cK - W) + i\tanh\left(\gamma\,(t - cx)\right)\right], \\ \delta_{||} \vec{X}\,(x) &= e^{iKx - i\sqrt{K^2-1}t}\,\vec{n}\left[\gamma\,(cK - W) + i\tanh\left(\gamma\,(t - cx)\right)\right], \end{aligned} \tag{4.7}$$

and $\delta X^1 = \delta X^2 = 0$ for both.[14] Fixing $t = 0$ and evaluating at large distance $x = \pm\frac{L}{2}$, they both become

$$\delta\vec{X}\left(\pm\frac{L}{2}\right) = e^{\pm iK\frac{L}{2} \pm i\delta_\pm} A_\pm,$$

where the phase shifts and amplitudes at the two ends are given by

$$\begin{aligned} \tan\left(\delta_\pm\right) &= \frac{-1 \mp \gamma\sqrt{1-K^2}}{\gamma c K}\,, \\ A_\pm &= \sqrt{(\gamma c K)^2 + \left(\gamma\sqrt{1-K^2} \pm 1\right)^2}\,. \end{aligned} \tag{4.8}$$

The next step would be to impose periodic boundary conditions on δX at $x = \pm\frac{L}{2}$. But here we encounter a problem, as the modes have different amplitudes at the two ends.[15] Instead we will demand only that the phases match at $x = \pm\frac{L}{2}$, and allow the amplitudes to be different. (We will discuss this further in the next section.) Then K has to obey

$$KL + \delta_+\,(K) + \delta_-\,(K) = K_n L,$$

where $K_n = \frac{2\pi n}{L}$ is still the discretised momentum of the vacuum solution. Taking L very large we can approximate K by

$$K = K_n - \frac{1}{L}\delta\,(K_n) + \mathcal{O}\left(\frac{1}{L^2}\right)$$

[14] To obtain this, note that K, W and k, ω are related by $K = -\gamma\left(ck + \sqrt{k^2-1}\right)$ and $W = -\gamma\left(k + c\sqrt{k^2-1}\right)$, from (2.14) and (2.4).

[15] Recall that the worldsheet velocity of the single spike is $1/c > 1$. Thus $(x, t) = (\pm L/2\,, 0)$ might be better thought of as points before and after the spike, rather than left and right of it. Consider instead points (x, t) with large $|t|$, for which both of the modes $\delta_\perp$ and $\delta_{||}$ in (4.7) become

$$\begin{aligned} \delta X &= e^{iKx - iWt}\left(\,\gamma\,(cK - W) + i\,\mathrm{sign}(t)\,\right) \\ &= e^{iKx + \sqrt{1-K^2}t}\left(\gamma c K - i\gamma\sqrt{1-K^2} + i\,\mathrm{sign}(t)\right)\,. \end{aligned}$$

In the second line we've chosen to focus on the growing mode $W = +i\sqrt{1-K^2}$. Averaging over x by taking the modulus, we get

$$|\delta X| = e^{\sqrt{1-K^2}t}\sqrt{(\gamma c K)^2 + \left(\mathrm{sign}(t) - \gamma\sqrt{1-K^2}\right)^2}\,.$$

This is an exponentially growing mode, with a step in it where the spike happens.

where $\delta(K) \equiv \delta_+(K) + \delta_-(K)$. Finally we can determine the imaginary correction to the energy of the spike from the four tachyonic modes, by putting $L \to \infty$:

$$\begin{aligned}\Delta E_{\text{spike}} &= 4\sum_K \frac{1}{2} W(K) \\ &\approx 4\frac{L}{2\pi}\int_{-1}^{1} dK\, \frac{1}{2} W\left(K - \frac{1}{L}\delta(K)\right) \quad \text{as } L \to \infty \\ &= \Delta E_{\text{hoop}} - i2\sqrt{\frac{1-c}{1+c}}.\end{aligned} \tag{4.9}$$

In the expression above, $\Delta E_{\text{hoop}} = iL/2$ is the correction (4.6) to the vacuum solution. Thus in the difference $\Delta E_{\text{spike}} - \Delta E_{\text{hoop}}$ the IR divergence from $L \to \infty$ is cancelled.

4.3 About these boundary conditions

We found that when $|K| < 1$ the amplitude of the mode (4.7) is different at large positive and negative x. This is the obstruction to imposing periodic boundary conditions, which we avoided by matching only the phases. One should not be surprised that we cannot impose periodic boundary conditions: they amount to gluing the string to itself after some large number of windings, or rather, gluing the vibrations on it to themselves, and this might not be allowed.

For the giant magnon, one has to glue a series of magnons together with $\sum_i p_i = 0$ to obtain a valid closed string solution. But is not clear that this is a condition on the allowed series of single spikes. It would tell you about periodicity of the spatial $X^i(x,t)$ under t, but say nothing about their behaviour at large $|x|$.

Here we consider a solution of two widely separated spikes with opposite velocities $\frac{1}{c}$ and $-\frac{1}{c}$, because for this choice we can impose honest boundary conditions. In this case we recover the twice the energy correction (4.9) obtained above, one for each spike. This justifies our use of these unusual boundary conditions.

Two spikes. As $x \to \pm\frac{L}{2}$, the amplitude of the mode (4.7) becomes $A_\pm$, given in (4.8). This formula is valid for $c > 0$; for $c < 0$ the sign $\pm$ is reversed, and we have instead $\left|\delta X_{c<0}(\pm\frac{L}{2}, 0)\right| = A_\mp$.

This immediately suggests the following way to impose consistent boundary conditions: take two spikes, far apart, with parameters c and $-c$. Each is in a box of length L, and we connect these together. That is, consider

$$X^\mu(x,t) = \begin{cases} X_{\text{spike}(c)}\left(x - \frac{L}{2}, t\right) & \text{for} \quad 0 < x < L\,, \\ X_{\text{spike}(-c)}\left(x - \frac{3L}{2}, t\right) & \qquad L < x < 2L \end{cases}$$

which is an approximate solution near $t = 0$. In fact it is a part of a scattering solution, since the two spikes have velocities $1/c$ and $-1/c$. It can be viewed as an excitation above a hoop of length $2L$.

Vibrations of this solution will be described by the same modes we have been using, and we again focus on the $|K| < 1$ sphere modes, which give the imaginary energy correction.

For the boundary condition at $x = L$, both modes δX have amplitude A_+, so matching them sets their phases equal there. And at $x = 0, 2L$ we can impose periodic boundary conditions, since both modes have amplitude A_- there. The resulting condition on the allowed K is simply

$$K = K_n - \frac{1}{2L}\delta_{(c)}\left(K_n\right) - \frac{1}{2L}\delta_{(-c)}\left(K_n\right) + \mathcal{O}\left(\frac{1}{L^2}\right),$$

where $K_n = \frac{2\pi n}{2L}$ are now the allowed wave numbers for the vacuum in length $2L$. This leads to energy correction

$$\Delta E = \Delta E_{\mathrm{spike}(c)} + \Delta E_{\mathrm{spike}(-c)}\,,$$

i.e. we obtain the sum of the corrections we calculated in (4.9) by imposing our phase-only boundary condition at $x = \pm\frac{L}{2}$. The finite piece (after subtracting the vacuum's ΔE_{hoop}) is twice the finite piece for one spike.

5. Conclusion

In this paper we determined the bosonic and fermionic modes of the single spike solution. Because there is a mismatch between the modes in these two sectors, both in number and in their masses, the spike cannot be supersymmetric. Some of the bosonic modes are tachyonic, showing that the single spike is unstable, like the relevant 'vacuum' solution which we referred to as the hoop.

We found that the Hamiltonian for small fluctuations of this vacuum is $\Delta - \Phi$. The winding Φ has replaced the angular momentum J found in the Hamiltonian for the magnon case, which is not surprising given that T-duality relates similar solutions in flat space. Using this result we performed a semi-classical calculation of the lifetime of the solution.

The dispersion relation for giant magnons (2.7) is periodic in p, which is the signature of discrete space. This is understood to be the position along a spin chain. One should not read the apparent lack of such periodicity in the single spike's case (2.8) as evidence against such discreteness. The recent paper [31] allows p outside our range $0 < p < \pi$, and finds that $\Delta - \Phi$ becomes periodic (their figure 1). However it is not clear that for the single spike this parameter p can still be interpreted as a spin-chain momentum.

It had been conjectured that the single spike is dual to an excitation of an anti-ferromagnetic spin chain. [28, 42] These have been various attempts to find an N-body description of the giant magnon, such as a Hubbard models, [67, 68] as was done for sine-gordon kinks. [69, 70] It is possible that this solution will be another test case for such a description.

The single spike is an excitation of an unstable vacuum state, the string wrapped around an equator of S^5. One can stabilise such loops of string by making them rotate in other planes. [71, 72] These can carry large angular momentum by being wound many times. It is possible that adding these extra angular momenta may stabilize the spike solution too, and it may be this object which has a more natural gauge theory dual.

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A. Fermionic zero modes

This appendix contains the exact analogue of the calculation done for the magnon in [57]. It has the virtue that it is easy to see why the single spike has twice as many modes as the magnon does; this was somewhat obscure in the non-zero mode calculations of section 3.2. But the result is identical to simply setting $\omega = 0$ there.

The zero modes are those with $\partial_u \Psi^I = 0$. Then the second-order equation (3.6) becomes

$$\left(\frac{1}{\gamma \cos\theta} \mathcal{D}_v \frac{1}{\gamma \cos\theta} \mathcal{D}_v + 1 \right) \Psi^1 = 0,$$

which factorises, and that is why the calculations are much easier than the non-zero modes.

This equation implies that $(\mathcal{D}_v - \eta\, i\gamma \cos\theta)\, \Psi^1 = 0$ with one of $\eta = \pm 1$, or pulling $(\rho_0 - \rho_1)$ out:

$$(\rho_0 - \rho_1) \left\{ \partial_v + \frac{1}{2} G \Gamma_{\phi\theta} + \eta\, i\gamma \cos\theta \right\} \Theta^1 = 0.$$

As for the non-zero modes, we first ignore the κ-symmetry projection and solve for Θ^1 alone. The matrix part of this equation involves only $\mathbf{1}$ and $\Gamma_{\phi\theta}$, which can be simultaneously diagonalised. Write the solution as

$$\Theta^1 = \Theta_+ + \Theta_- = f_+(v) U_+ + f_-(v) U_- \,,$$

where the spinors $U_\pm$ (and so $\Theta_\pm$) are $\Gamma_{\phi\theta}$ eigenvectors, with eigenvalues $\pm i$ respectively. All that is left to solve is

$$\left\{ \partial_v \pm \frac{i}{2} G + \eta\, i\gamma \cos\theta \right\} f_\pm(v) = 0.$$

The solutions are pure phase,

$$f_\pm(v) = e^{\pm i\chi} e^{i\eta\chi_2} \qquad \text{where} \quad e^{i\chi} = \left(\frac{\sinh v + ic}{\sinh v - ic} \right)^{1/4} \sqrt{\tanh v + i \,\mathrm{sech}\, v},$$
$$e^{\pm i\chi_2} = \mathrm{sech}\, v \pm i \tanh v.$$

The difference between these solutions and the giant magnon's ones [57] is that instead of a modulating factor $\mathrm{sech}\, u$, we get an extra phase $e^{i\eta\chi_2}$. It is this modulating factor which makes one solution normalisable, and allows Minahan to reject the other sign of η for producing a solution which diverges at large u. But in our case both signs lead to non-normalisable solutions. The general solution is a linear combination of the $\eta = +1$ and $\eta = -1$ cases:

$$\Psi^1 = -i(\rho_0 - \rho_1)\frac{1}{\sqrt{1-c}}\sum_{\pm} e^{\pm i\chi} \sum_{\eta} e^{i\eta\chi_2} U_{\pm}^{\eta}\,.$$

(We've smuggled in a factor of $\sqrt{1-c}$ for reasons of aesthetic balance between between Ψ^1 and Ψ^2.) Writing out $(\rho_0 - \rho_1)$ and using the identity $e^{\pm 2i\chi} = (p-r) \mp i(q-s)$, we obtain:

$$\Psi^1 = \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right]\left(\operatorname{sech} v\, U_0 + \tanh v\, \tilde{U}_0\right)$$

where we've combined the arbitrary spinors $U_{\pm}^{\eta}$ into

$$U_0 = -\left(U_+^{\eta=1} + U_-^{\eta=1}\right) - \left(U_+^{\eta=-1} + U_-^{\eta=-1}\right),$$
$$\tilde{U}_0 = -i\left(U_+^{\eta=1} + U_-^{\eta=1}\right) + i\left(U_+^{\eta=-1} + U_-^{\eta=-1}\right).$$

The reason for this choice is that the Majorana condition $\Psi^* = \Psi$ now simply requires that U_0 and $\tilde{U}_0$ be themselves Majorana spinors. (The Γ-matrices are all imaginary, thus $\Gamma_{\phi\theta}$ is real.)

Having found Ψ^1, we immediately have Ψ^2 as an operator acting on it, from (3.5), with no further choices to make. We can write:

$$\Psi^2 = \frac{\Gamma_\star\Gamma_\theta}{\sqrt{1+c}}\left[\Gamma_0\left(\cos\tilde{\chi} + \Gamma_{\phi\theta}\sin\tilde{\chi}\right) - \Gamma_\phi\left(\cos\tilde{\chi} - \Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right]\left(\operatorname{sech} v\, \tilde{U}_0 - \tanh v\, U_0\right)$$

where as before $e^{i\tilde{\chi}} = e^{-i\chi + i\chi_2}$, and $(r \pm i\, s) = \pm i\gamma\cos\theta e^{\pm i(\tilde{\chi}-\chi)}$ was used.

Comparing these zero modes with the non-zero modes (3.13) and (3.15), it is clear that they are simply the $\omega = 0$ case of the latter (i.e. $\alpha = \beta = 0$). This is different from the supersymmetric giant magnon case, where the massive non-zero modes of [53] do not connect to the zero modes of [57].

Let us pause to count these modes: the four spinors $U_{\pm}^{\eta}$ are $\Gamma_{\phi\theta}$-eigenspinors, thus have 16 complex components each. They must be Weyl spinors, i.e. Γ_{11}-eigenspinors, which cuts the number in half. Requiring U_0 and $\tilde{U}_0$ to be Majorana cuts it in half again, to 16 components in total. This is the same number we would count by setting $\omega = 0$ in the non-zero modes above, as must be the case: U_0 is the same spinor. At this stage [57] had 8 complex components. The argument below cuts it by another factor of 2 in both cases.

Slow-motion. In [57], Minahan uses an argument which runs as follows: regard the spinors U_0 and $\tilde{U}_0$ as a moduli of the solution, and allow them to become time-dependent, $\partial_u U \neq 0$. Substituting a zero mode into the action will give zero, but this 'slowly-moving' mode needn't do so. The zero modes whose slowly-moving cousins give a non-zero action are 'real' zero modes, the others pure gauge. [73]

When substituting the slowly-moving mode $\Theta = \sum F(v)U(u)$ into the Lagrangian, the equations of motion force everything except the ∂_u terms to vanish, leaving

$$\mathcal{L}_F = -i\gamma(1-c)\,\overline{\Theta}^1(\rho_0-\rho_1)\partial_u\Theta^1 + i\gamma(1+c)\,\overline{\Theta}^2(\rho_0+\rho_1)\partial_u\Theta^2.$$

(As usual $\overline{\Theta} = \Theta^\dagger\Gamma_0$.) Using the identities $2\Gamma_0(\rho_0\pm\rho_1) = -(\rho_0\pm\rho_1)^\dagger(\rho_0\pm\rho_1)$ this becomes

$$\mathcal{L}_F = i\gamma\frac{1-c}{2}\Psi^{1\dagger}\partial_u\Psi^1 - i\gamma\frac{1+c}{2}\Psi^{2\dagger}\partial_u\Psi^2.$$

Now plug in Ψ^I from above, to obtain

$$\begin{aligned}\mathcal{L}_F = \frac{i\gamma}{2}&\left[(\Gamma_0-\Gamma_\phi)\left(\operatorname{sech} v\, U_0 + \tanh v\,\tilde{U}_0\right)\right]^\dagger \partial_u\left[(\Gamma_0-\Gamma_\phi)\left(\operatorname{sech} v\, U_0 + \tanh v\,\tilde{U}_0\right)\right]\\ -\frac{i\gamma}{2}&\left[(\Gamma_0-\Gamma_\phi)\left(\operatorname{sech} v\,\tilde{U}_0 - \tanh v\, U_0\right)\right]^\dagger \partial_u\left[(\Gamma_0-\Gamma_\phi)\left(\operatorname{sech} v\,\tilde{U}_0 - \tanh v\, U_0\right)\right].\end{aligned}$$

Both U_0 and $\tilde{U}_0$ always appear acted on by $(\Gamma_0-\Gamma_\phi)$. Thus only those modes satisfying $(\Gamma_0+\Gamma_\phi)U_0 = 0$ and $(\Gamma_0+\Gamma_\phi)\tilde{U}_0 = 0$ contribute. The situation is identical to the magnon case in that only half of the modes appearing in Ψ^I contribute here ('are real,' meaning true, zero modes). But since there are two constant Majorana-Weyl spinors U_0 and $\tilde{U}_0$ here, instead of only one for the giant magnon, there are twice as many modes: 8 instead of 4 complex degrees of freedom. Thus for both the non-zero modes and the zero modes, in the fermionic sector, we find twice as many as in the giant magnon case.

B. Hamiltonian formulation and energy corrections

B.1 Quadratic 2-dimensional hamiltonian

Starting from the Lagrangian for the fluctuations (4.3), we find its quadratic part to be (up to factors of λ):

$$\tilde{\mathcal{L}}^2 = \frac{1}{2}\left(-\partial^a\tilde{\tau}\partial_a\tilde{\tau} + \partial^a\tilde{\eta}_k\partial_a\tilde{\eta}_k + \partial^a\tilde{\phi}\partial_a\tilde{\phi} + \partial^a\tilde{\theta}_s\partial_a\tilde{\theta}_s + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right).$$

By determining the conjugate momenta for each of the fluctuation fields, $\tilde{\Pi}_\mu = \frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_0\tilde{X}^\mu)}$, we find

$$\begin{aligned}\tilde{\Pi}_{\tilde{\tau}} &= \partial_0\tilde{\tau}\\ \tilde{\Pi}_{\tilde{X}^\mu} &= -\partial_0\tilde{X}^\mu \quad \text{for } \tilde{X}^\mu = \tilde{\eta}_k, \tilde{\theta}_s, \tilde{\phi}.\end{aligned}$$

From these we can construct the corresponding Hamiltonian density in the usual way, obtaining

$$\tilde{\mathcal{H}}^2 = \frac{1}{2}\left(-\tilde{\Pi}^2_{\tilde{\tau}} + \tilde{\Pi}^2_{\tilde{\phi}} + \tilde{\Pi}^2_{\tilde{\eta}_k} + \tilde{\Pi}^2_{\tilde{\theta}_s} + \tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right).$$

We want to check that the quantity $\Delta-\Phi$ is just this Hamiltonian. To do so we start by determining the Hamiltonian corresponding to the original bosonic Lagrangian (4.1), and expand it in fluctuations. The conjugate momenta for the fields are given by $\Pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 X^\mu)}$

where $X^\mu = \tau, \phi, \eta_k, \theta_s$. To find the Hamiltonian for the fluctuations, we expand the fields as in (4.2), as well as the momenta:

$$\Pi_\mu = \Pi_\mu^{cl} + \lambda^{-\frac{1}{4}}\tilde{\Pi}_\mu \quad ; \qquad X^\mu = X^\mu_{cl} + \lambda^{-\frac{1}{4}}\tilde{X}^\mu ,$$

where the classical values of the fields are $\Pi_\tau^{cl} = 1$, $\Pi^{cl}_{X^\mu \neq \tau} = 0$, $\tau_{cl} = t$, $\phi_{cl} = x$ and all other fields are zero. The expansion of the Hamiltonian then gives:

$$\begin{aligned}\mathcal{H}_b = &\frac{1}{2\sqrt{\lambda}}\left(-\tilde{\Pi}^2_{\tilde{\tau}} + \tilde{\Pi}^2_{\tilde{\phi}} + \tilde{\Pi}^2_{\tilde{\eta}_k} + \tilde{\Pi}^2_{\tilde{\theta}_s} - (\partial_1\tilde{\tau})^2 + \left(\partial_1\tilde{\phi}\right)^2 + (\partial_1\tilde{\eta}_k)^2 + \left(\partial_1\tilde{\theta}_s\right)^2\right) + \\ &+\frac{1}{2\sqrt{\lambda}}\left(\tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\right) - \frac{1}{\lambda^{\frac{1}{4}}}\left(\tilde{\Pi}_{\tilde{\tau}} - \left(\partial_1\tilde{\phi}\right)\right) + \mathcal{O}\left(\frac{1}{\lambda}\right).\end{aligned}$$

The Virasoro constraint (4.4) is equivalent to setting $\mathcal{H}_b = 0$.

It is easy to check that $\Delta - \Phi$ can be written in terms of the fields and conjugate momenta as

$$\Delta - \Phi = \frac{\sqrt{\lambda}}{2\pi}\int dx \left(\frac{1}{\lambda^{\frac{1}{4}}}\left(\tilde{\Pi}_{\tilde{\tau}} - \left(\partial_1\tilde{\phi}\right)\right)\right).$$

By using the Virasoro constraint in the form $\mathcal{H}_b = 0$, we finally find

$$\begin{aligned}\Delta - \Phi = \int \frac{dx}{2\pi}\Big(&-\tilde{\Pi}^2_{\tilde{\tau}} + \tilde{\Pi}^2_{\tilde{\phi}} + \tilde{\Pi}^2_{\tilde{\eta}_k} + \tilde{\Pi}^2_{\tilde{\theta}_s} - (\partial_1\tilde{\tau})^2 + \left(\partial_1\tilde{\phi}\right)^2 + (\partial_1\tilde{\eta}_k)^2 + \left(\partial_1\tilde{\theta}_s\right)^2 \\ &+\tilde{\eta}_k\tilde{\eta}_k - \tilde{\theta}_s\tilde{\theta}_s\Big),\end{aligned}$$

which returns the expected expression, when we drop the gauge fluctuations.

B.2 Modes for the hoop (vacuum) solution

It is simple to solve the equations of motion from the bosonic Lagrangian $\mathcal{L}_B$ (4.3) in order to determine the modes for the hoop solution. The transverse modes are

$$\begin{aligned}\tilde{\eta}_k(x,t) &= e^{iKx - iWt} f_k(K), \qquad W^2 = K^2 + 1\,, \\ \tilde{\theta}_s(x,t) &= e^{iKx - iWt} g_s(K), \qquad W^2 = K^2 - 1\,,\end{aligned}$$

i.e $m^2 = 1$ in the AdS directions, and $m^2 = -1$ on the sphere, the same masses as for the single spike's modes. The longitudinal modes are massless:

$$\begin{aligned}\tilde{\tau}(x,t) &= e^{iKx - i|K|t} f(K), \\ \tilde{\phi}(x,t) &= e^{iKx - i|K|t} g(K).\end{aligned}$$

The same modes can also be obtained from those for the single spike, by going far away from the spike itself. The sphere modes $\delta_\perp$ (2.12) and $\delta_{||}$ (2.13) of section 2.4 become these simple ones $\tilde{\theta}_s$ in the limit $v \to \infty$, and the AdS modes are identical. The $\tilde{\phi}$ mode is the $v \to \infty$ limit of δ_r (2.11), now more obviously pure gauge. We did not write down the analogue of the $\tilde{\tau}$ mode (among the spike's non-zero modes) as we were focusing on the spatial part, but this too is pure gauge.

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Performing the same limit $v \to \infty$ for the fermionic modes (3.13) and (3.15) leaves the following modes for the hoop:

$$\begin{aligned}\Psi^1 &= \frac{i}{\sqrt{1-c}}\left[\Gamma_0\left(\cos\chi + \Gamma_{\phi\theta}\sin\chi\right) - \Gamma_\phi\left(\cos\chi - \Gamma_{\phi\theta}\sin\chi\right)\right] \\ &\qquad \times \left(\cos\beta\,\tilde{U}_0 + \sin\beta\,\Gamma_{\phi\theta}\tilde{U}_1\right), \\ \Psi^2 &= \frac{-1}{\sqrt{1+c}}\Gamma_*\Gamma_\theta\left[\Gamma_0\left(\cos\tilde{\chi} + \Gamma_{\phi\theta}\sin\tilde{\chi}\right) - \Gamma_\phi\left(\cos\tilde{\chi} - \Gamma_{\phi\theta}\sin\tilde{\chi}\right)\right] \\ &\qquad \times \frac{1}{1+4\omega^2}\left(\cos\tilde{\beta}\,U_0 + \sin\tilde{\beta}\,\Gamma_{\phi\theta}U_1\right).\end{aligned}$$

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